Real Analysis qualification exam

Note: all statements require proofs. You can make references to standard theorems from the course, but you need to specify which exact standard fact you are referring to.

1. Let $E \subset [0,1]^2$ be a Lebesgue measurable subset of positive measure. Show that there exist $x, y \in E$, $x \neq y$, such that $x - y \in \mathbb{Q}^2$.

2. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) < +\infty$, and $f_n \colon X \to [0, +\infty)$ be measurable functions. Show that $f_n \to 0$ in measure if and only if

$$\int_X \frac{f_n}{1+f_n}\, d\mu \to 0, \quad \text{as } n \to \infty.$$

   Is the assumption $\mu(X) < +\infty$ necessary?

3. Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $\mu(X) = 1$, and $f, g \colon X \to [0, +\infty]$ are measurable functions satisfying $f(x)g(x) \geqslant 1$ for all $x \in X$. Show that

$$\int_X f\, d\mu \cdot \int_X g\, d\mu \geqslant 1.$$

4. Let $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ be measure spaces, $\mu_1(X_1), \mu_2(X_2) < +\infty$. Suppose that ν_1, ν_2 are finite measures defined on $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ respectively, and ν_j is absolutely continuous with respect to μ_j, $j = 1, 2$. Show that the product measure $\nu_1 \times \nu_2$ is absolutely continuous with respect to the product measure $\mu_1 \times \mu_2$.

   Let $f_j = \frac{d\nu_j}{d\mu_j}$, $j = 1, 2$, be the densities of ν_j with respect to μ_j. Find the density $\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}$ in terms of f_j.

5. Let $f \colon [0,1] \to [0,1]$ be a continuous strictly increasing function such that it maps any (Lebesgue) null set into a null set. Show that f is absolutely continuous.